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# DESIGN SPECTRA FOR USE IN PROBABILITY-BASED DESIGN WITH EQUIVALENT LINEARIZATION TECHNIQUE

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## ABSTRACT

This paper first investigates the appropriateness of the use of a uniform hazard spectrum (UHS) along with an equivalent linearization technique as a design spectrum to estimate the response of an inelastic oscillator. There exists some correlation among the spectral responses of elastic oscillators to a ground motion, but such correlation is not considered in a UHS. An approximation method is proposed to estimate the probability distribution function of the response using the probabilistic characteristics of an elastic response spectrum and Capacity-Spectrum method, in which the inelastic response is estimated by finding the intersection of the inelastic demand spectrum and the capacity spectrum. The accuracy of the proposed method is investigated using numerical examples.

## Introduction

Predictors of seismic structural demands (such as inter-story drift angles) that are less time-consuming than nonlinear dynamic analysis (NDA) have proven useful for structural performance assessment and for design. Several techniques have been proposed using the results of a nonlinear static pushover analysis (e.g., Luco 2002; Chopra & Goel 2002; Yamanaka, et al 2003; Mori, et al, 2006). These techniques often use the maximum response computed via NDA of the inelastic oscillator that is "equivalent" to the original frame. In practice, it is desirable to estimate the response approximately via a simpler method such as a design response spectrum at the site and R-factor or, a little more accurately, the spectrum and an equivalent linearization technique.

In reliability-based seismic design of a structure, design spectra are being developed based on the probabilistic approach, and a uniform hazard spectrum (UHS) is often used. A UHS is developed by plotting the response with the same (i.e. uniform) exceedance probability of a suit of elastic oscillators with natural period different from one another, and accordingly, does not represent any specific ground motion (Abrahamson 2006). Although there exits some correlation among the spectral responses of oscillators to a ground motion (e.g. Baker & Cornell 2006), such correlation is not considered in a UHS. Hence, it might not be suitable to use a UHS to estimate the inelastic response via an equivalent linearization technique, in which the response at the equivalent natural period is also used. As it is implicitly assumed in a use of a UHS that the

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response of a suite of the elastic oscillators is perfectly correlated, the response would be overly estimated when very rare event is considered.

This paper first investigates the appropriateness of the use of a UHS, i.e., the assumption of the perfect correlation, along with an equivalent linearization technique as a design spectrum to estimate the response of an inelastic oscillator. The results are compared with those of Monte Carlo simulation considering correlation among the responses of a suit of elastic oscillators with natural period different from one another. Then it proposes an approximation method to estimate the probability distribution function (CDF) of the response of an inelastic oscillator via an equivalent linearization technique using the probabilistic characteristics of an elastic response spectrum. In the Capacity-Spectrum method (Freeman 1978), which is employed in the current Japanese seismic provisions, the inelastic response is estimated by finding the intersection of the inelastic demand spectrum and the capacity spectrum. When the uncertainty in the demand spectrum is to be taken into account, the problem could be considered as a first-passage problem in a standard normal stochastic process with a time-varying threshold. An approximation method is then proposed considering the probability that the demand spectrum is above the capacity spectrum at two or three spectral periods including the elastic natural period of the oscillator. The accuracy of the method is investigated using numerical examples.

## Equivalent Linearization Technique and Capacity-Spectrum Method

In an equivalent linearization technique, the maximum displacement of an inelastic oscillator with elastic natural period, $T_1$, and damping factor, $h_1$, to a ground motion is approximated with the maximum displacement of an elastic oscillator with the equivalent natural period, $T_{eq}$, and the equivalent damping factor, $h_{eq}$, to the ground motion as,

$$S_D^I(T_1;h_1) \approx S_D^E(T_{eq};h_{eq}) \tag{1}$$

in which $S_D(T;h)$ is the spectral displacement of an oscillator with natural period, $T$, and damping factor, $h$, and the superscripts $E$ and $I$ represent the elastic response and inelastic response, respectively.

Generally, $T_{eq}$ and $h_{eq}$ are estimated as the function of the maximum ductility factor of the inelastic oscillator, $\mu$, which is defined as the ratio of the maximum displacement to the yield displacement. Several techniques have been proposed (eg., Iwan 1980, Shimazaki 1999), and among them, the following $T_{eq}$ and $h_{eq}$ proposed by Shimazaki is considered in this paper.

$$T_{eq} = T_1 \cdot \sqrt{\mu} \tag{2}$$

$$h_{eq} = 0.25 \cdot \left(1 - \frac{1}{\sqrt{\mu}}\right) + h_1 \tag{3}$$

Applying Capacity Spectrum method, the inelastic displacement can be estimated graphically as the intersection of capacity spectrum and demand spectrum. In order to take into account the effect of $h_{eq}$, the demand spectrum must be adjusted by multiplying damping reduction factor $F_h(h_{eq})$. Since $F_h(h_{eq})$ is the function of the unknown value $\mu$, it generally requires iterative procedure.

On the contrary, the response can be estimated directly by considering the Demand and Capacity spectra in an ordinal $T$ - $S_D$ coordinate rather than an $S_D$ - $S_A$ coordinate as follows (see Fig.1, Mori and Maruyama, 2007):

Divided by the yield displacement of the inelastic oscillator, the $S_D$ axis can be transformed linearly into the axis of the maximum ductility factor, $\mu$. The $T$ axis can also be expressed in terms of $\mu$, since $T_{eq}$ is a function of $\mu$ as expressed by Eq.2. Then the Capacity Spectrum can be obtained as the one-to-one line in the $\mu$ - $\mu$ coordinate.

The inelastic demand spectrum can be obtained as the product of the elastic demand spectrum and a damping reduction factor, $F_h(h_{eq})$, in which $h_{eq}$ is also a function of $\mu$ (see Eq.3). Then, the maximum displacement of the inelastic oscillator is obtained as the intersection (point $A$ in Fig.1) of the capacity spectrum and the inelastic demand spectrum ($S_D^E(T_{eq};h_{eq}) \cdot F_h(h_{eq})$ in Fig.1). Note that $T_{eq}$ can also be obtained as the intersection of capacity spectrum multiplied by $1/F_h(h_{eq})$ ($g(T)$ in Fig.1) and the elastic demand spectrum (point $A'$ in Fig.1). Then the maximum displacement can be evaluated using Eq.2.

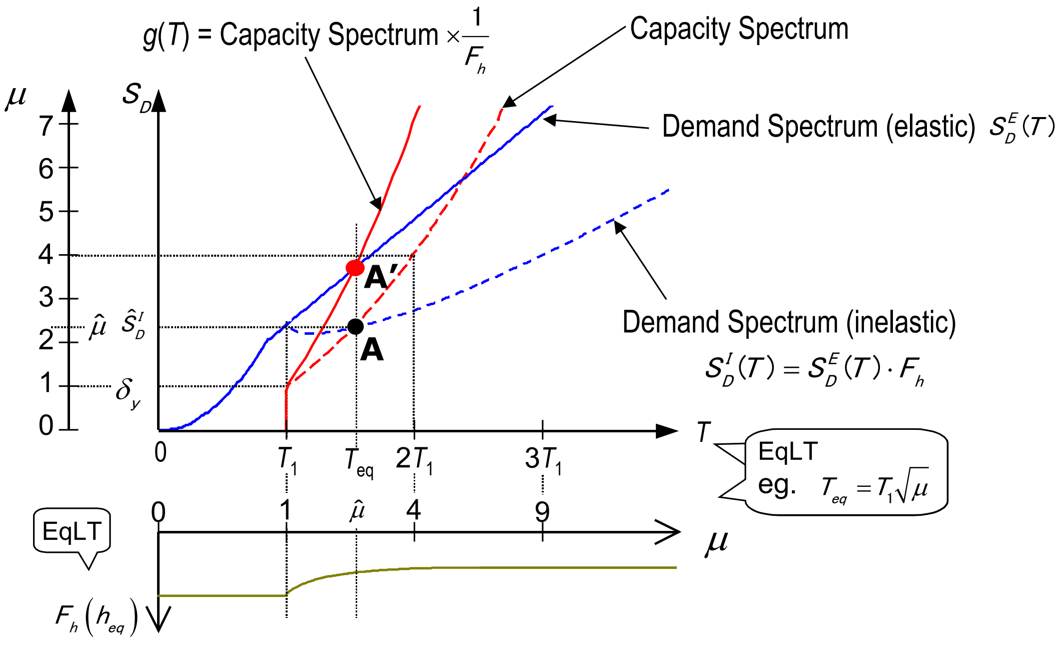


Figure 1. Capacity-Spectrum Method in $T$ - $S_D$ coordinate

## CDF of $S_D^I(T)$ by Monte Carlo Simulation

### Probabilistic Model of Seismic Hazard

The basic information of seismic hazard considered in this paper is the probability distribution of the maximum spectral displacement of an elastic oscillator in $n$ years, $S_{Dn}^E(T;h)$, and auto-correlation function of the spectrum. Such information can be obtained through a seismic hazard analysis, which generally takes the following steps.

(1) Simulate the occurrence of earthquakes at the faults which could cause a strong ground motion at a construction site for the next $n$ years.
(2) Evaluate the response spectrum at the site for each earthquake using attenuation formula. The variability and the auto-correlation functions of a response spectrum are considered by multiplying an random variable with median equal to 1.0.
(3) Take the maximum response among the earthquakes at each natural period as the maximum response of the sample of $n$ years.
(4) By repeating (1)-(3) many times obtain the probability distribution function and the auto-correlation function of $S_{Dn}^E(T;h)$.

It is assumed here that a response spectrum of a ground motion has the auto-correlation functions of a response spectrum, $K_S(\xi)$, proposed by Baker and Cornell (2006) (see Fig.2) for each ground motion. It should be noted that although $K_S(\xi)$ is in general not a function of the difference between two spectral periods, $T_i$ and $T_j$; however, in most of the proposed models including Baker and Cornell's model it is only the function of $\xi = \left|\log(T_i) - \log(T_j)\right|$.

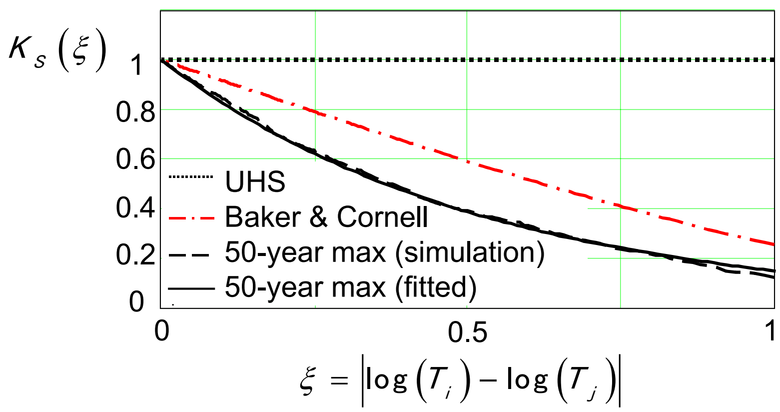


Figure 2. Auto-correlation function of response spectrum, $K_S(\xi)$

### Capacity Spectrum

Using equivalent linearization technique proposed by Shimazaki (1999) (see Eq.2 and 3),

$g(T)$ (see Fig.1) of an inelastic oscillator with the mass equal to unity and an elasto-plastic backbone curve can be expressed as,

$$g(T)=\frac{\delta_y\cdot\mu}{F_h(h_{eq})}=\frac{\delta_y}{F_h(h_{eq})}\left(\frac{T}{T_1}\right)^2=\frac{9.8\cdot C_y}{4\pi\cdot F_h(h_{eq})}T^2\ \ ;\ \ T\geq T_1 \tag{4}$$

in which $C_y$ and $\delta_y$ are the yield base shear coefficient and the yield displacement of the oscillator, respectively, and 9.8(m/s) is the gravity acceleration. The following damping reduction factor proposed by Kasai et al (2003) is used in this paper.

$$F_h(h)=\begin{cases}\left(\sqrt{D(h)}-1\right)\cdot(5\cdot T)+1 & ;\text{if } 0\leq T\leq 0.2\\ D(h) & ;\text{if } 0.2\leq T\leq 2\\ D(h)\cdot\left\{\dfrac{\sqrt{h/h_0}\cdot(T-2)}{40}+1\right\} & ;\text{if } 2\leq T\leq 8\end{cases} \tag{5}$$

in which

$$D(h)=\sqrt{\frac{1+25\cdot h_1}{1+25\cdot h}} \tag{6}$$

**CDF of $S_D^I(T)$ Using UHS**

In order to investigate the appropriateness of the use of a UHS along with an equivalent linearization technique, 8,000 samples of 50-year seismic activities and the response spectrum for each ground motion at Nagoya, Japan are generated. It is assumed that the random variable described in Step (2) in the previous section is lognormally distributed with coefficient of variation (c.o.v) equal to 0.5. The statistics of the 50-year maximum spectral displacement of an elastic oscillator with $h=0.05$, $S_{D50}^E(T;h=0.05)$, is illustrated in Fig.3. The auto-correlation function of the 50-year maximum spectrum is also illustrated in Fig.2. Note that the correlation of the 50-year maximum response spectrum is lower than that of a ground motion. This is because of the possibility that one of the earthquakes could cause the 50-year maximum response at one natural period while the other could cause the maximum response at the other natural period.

The exceedance probability (complementary CDF) in $n$ years of the maximum displacement of the inelastic oscillator is illustrated in Fig. 4 assuming that (a) the auto-correlation model by Baker and Cornell (see Fig.2) and (b) the perfectly correlated model, which corresponds to a UHS. It is assumed that the damping factor and the yield base shear coefficient of the inelastic oscillator equal 0.05 and 0.3, respectively and that the natural period equals 0.5, 1.0, 1.5, or 2.0 seconds. The maximum displacement of the oscillator is estimated using Capacity spectrum method in $T-S_D$ coordinate for each of 8,000 samples of 50-year seismic activities. The maximum response is overly estimated using a UHS, and thus, the auto-correlation of spectral response, $K_S(\xi)$, should be taken into account appropriately when the equivalent linearization technique is applied for estimating the $n$-year maximum displacement of an inelastic oscillator.

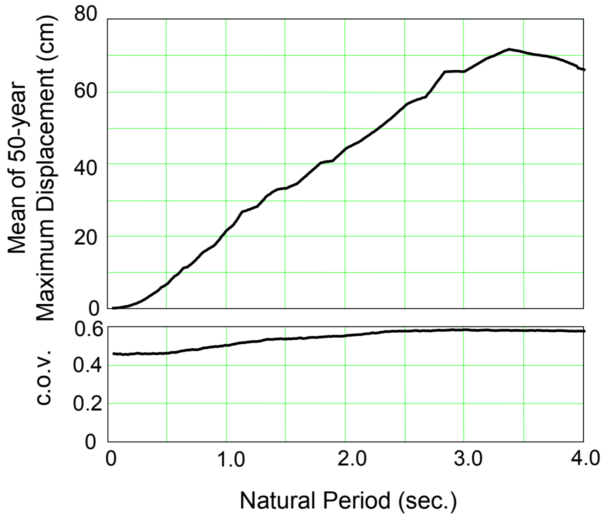


Figure 3. Statistics of 50-year maximum acceleration response spectrum

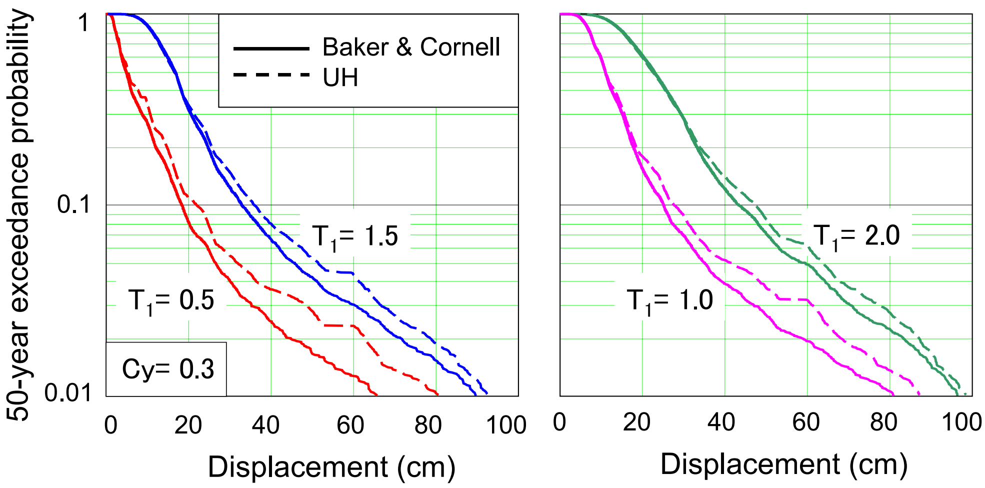


Figure 4. 50-year maximum displacement of inelastic oscillator

## Approximation Method for CDF of $S_D^I(T)$

As discussed in the previous section, the maximum displacement of an inelastic oscillator would be overly estimated using of a UHS along with equivalent linearization technique. Although it can be estimated by Monte Carlo simulation, it requires quite a lot of computational effort.

In the Capacity Spectrum method, the event that the equivalent natural period, $T_{eq}$, is longer than $t_{eq}$ corresponds to the event that $S_{D50}^{E}(T;h)$ is always above $g(T)$ within the range of $\left(T_1, t_{eq}\right)$ (the hatched area in Fig.5 (a)).

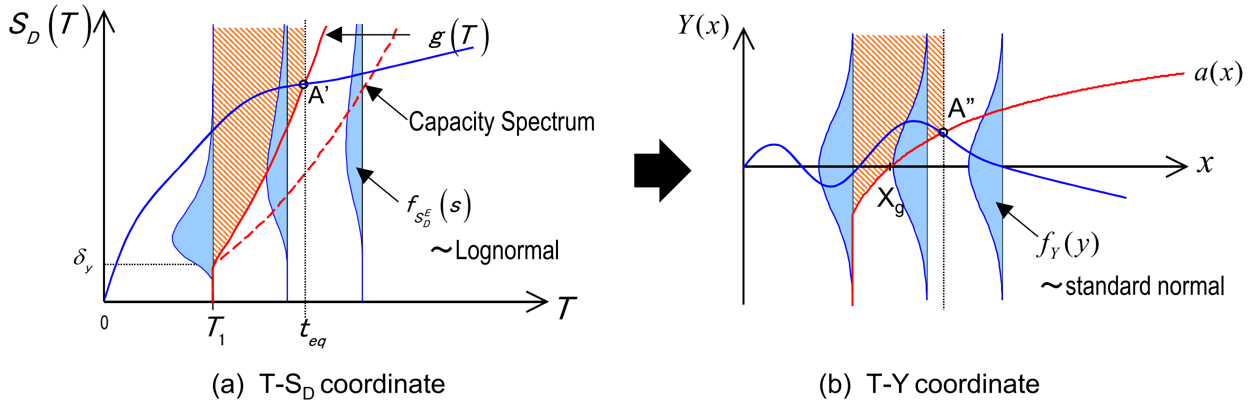


Figure 5. Schematic illustration of Capacity Spectrum method for standard normal stochastic process

By transforming $S_{D50}^{E}(T;h)$, which is assumed to be lognormally distributed based on the hazard analysis, into standard normal stochastic process, $Y(T)$, by Eq.7, the function $g(T)$ and the hatched area in Fig.4(a) is transformed to the function $a(T)$ and the hatched area in $T-y$ coordinate in Fig.4(b), respectively.

$$Y(T) = \frac{\ln(S_{D50}^{E}(T;h)) - \mu_{\ln\left(S_{D50}^{E}(T)\right)}}{\sigma_{\ln\left(S_{D50}^{E}(T)\right)}} \tag{7}$$

in which $\mu_{\ln\left(S_{D50}^{E}(T)\right)}$ and $\sigma_{\ln\left(S_{D50}^{E}(T)\right)}$ are the mean and standard deviation of $\ln(S_{D50}^{E}(T;h))$, respectively. The auto-correlation function of $Y(T)$, $K_Y(\xi)$, is expressed as (Der Kiureghian and Liu, 1985),

$$K_Y(\xi) = \frac{\ln\left(1 + K_S(\xi)\cdot V_{SD50(T_i)}\cdot V_{SD50(T_j)}\right)}{\sqrt{\ln\left(1+V_{SD50(T_i)}^2\right)}\sqrt{\ln\left(1+V_{SD50(T_j)}^2\right)}} \tag{8}$$

in which $V_{SD50(T_i)}$ is the c.o.v. of the $S_{D50}^{E}(T_i;h)$. Then the problem of estimating the probability that the stochastic process $S_{D50}^{E}(T;h)$ stays above the threshold $g(T)$ within the range of $\left[T_1, t_{eq}\right]$ can be considered as the first passage problem of a standard normal stochastic process crossing the threshold $a(T)$ downward.

The first passage problem of a standard normal stationary stochastic process has been well studied. Most of the studies have been devoted to estimate the probability of rare events, for which the threshold is relatively far from the $T$-coordinate. However, as seen in Fig.5, in the current problem the threshold crosses the $T$-coordinate, and the existing theories are not applicable. Here, a simple approximation method is proposed based on the probability that the spectrum is above the threshold $a(T)$ at $T_1$, $t_{eq}$, and the geometrical mean of $T_1$ and $t_{eq}$.

Consider two standard normal random variables $X$ and $Y$ with correlation coefficient $\rho$. Given $Y = y_1$, $X$ is a normal random variable with mean value $\mu_X = \rho \cdot y_1$ and standard deviation $\sigma_X = \sqrt{1-\rho^2}$. Then, given $Y(T_1) = y_1$, the conditional probability that the stochastic process $Y(T)$ is above the threshold $a(T)$ at the natural period equal to $t_{eq}$ can be expressed as,

$$P\left[Y(t_{eq}) \geq a(t_{eq}) \mid Y(T_1) = y_1\right] = \begin{cases} 1-\Phi\left(\dfrac{a(t_{eq})-\rho_{1,eq}\cdot y_1}{\sqrt{1-\rho_{1,eq}{}^2}}\right) & ;\ y_1 \geq a(T_1) \\ 0 & ;\ y_1 < a(T_1) \end{cases} \tag{9}$$

in which $\Phi(\ )$ is the standard normal probability distribution function and $\rho_{1,eq}$ is the correlation coefficient of $S_{D50}^{E}(T_i;h)$ at the natural periods $T_1$ and $t_{eq}$.

By the theorem of total probability, the probability that $Y(T)$ is above the threshold $a(T)$ at the natural period equal to $T_1$ and $t_{eq}$ can be expressed as,

$$P\left[Y(T_1) \geq a(T_1) \cap Y(t_{eq}) \geq a(t_{eq})\right] = \int_{a(T_1)}^{-\infty}\left(1-\Phi\left(\frac{a(t_{eq})-\rho_{1,eq}\cdot y_1}{\sqrt{1-\rho_{1,eq}{}^2}}\right)\right)\phi(y_1)dy_1 \tag{10}$$

in which $\phi(\ )$ is the standard normal probability density function.

In Eq.10, the possibility that $Y(T)$ crosses the threshold $a(T)$ downward within the interval $\left(T_1, t_{eq}\right)$ is not considered, and accordingly the response of an inelastic oscillator could be overly estimated by this equation. Here, the possibility that $Y(T)$ is above the threshold $a(T)$ at one more point within the interval $\left(T_1, t_{eq}\right)$ is considered. As the auto-correlation function of $S_{D50}^{E}(T_i;h)$ is only the function of $\xi = \left|\log\left(T_i\right)-\log\left(T_j\right)\right|$ (see Fig.2), the geometric mean of $T_1$ and $t_{eq}$ is selected as the third point. Then the following semi-empirical formula is proposed as an approximation method.

$$P\left[T_{eq} \geq t_{eq}\right] \approx P\left[Y(T_1) \geq a(T_1) \cap Y(t_m) \geq a(t_m) \cap Y(t_{eq}) \geq a(t_{eq})\right]$$

$$= \int_{a(T_1)}^{\infty} P\left[Y(t_m) \geq a(t_m) \cap Y(t_{eq}) \geq a(t_{eq}) \mid Y(T_1) = y_1\right] \cdot \phi(y_1) dy_1$$

$$\approx \int_{a(T_1)}^{\infty} \left\{1 - \Phi\left(\frac{a(t_m) - \rho_{1,m} \cdot y}{\sqrt{1 - \rho_{1,m}^2}}\right)\right\}\left\{1 - \Phi\left(\frac{a(t_{eq}) - \rho_{1,eq} \cdot y}{\sqrt{1 - \rho_{1,eq}^2}}\right)\right\} \cdot \phi(y) dy \qquad (11)$$

in which $\rho_{1,m}$ is the correlation coefficient of the spectrum at natural periods $T_1$ and $t_m$, and $t_m = \sqrt{T_1 \cdot t_{eq}}$. The probability distribution function of the maximum displacement of an inelastic oscillator can be estimated using Eq.2 and Eq.11.

**Accuracy of Proposed Method**

Fig.6 illustrates the exceedance probability of the spectral displacement of an inelastic oscillator with the elastic natural period, $T_1$, equal to (a) 0.5 sec., (b) 1.0 sec., (c) 1.5 sec., and (d) 2.0 sec., respectively, evaluated by Eq.10, Eq.11, and Monte Carlo simulation. Eq.10 provides overly pessimistic estimates of the response as expected. On the contrary, Eq.11 provides optimistic estimates. Further investigation is required for more accurate estimates which would exist between Eqs.10 and 11.

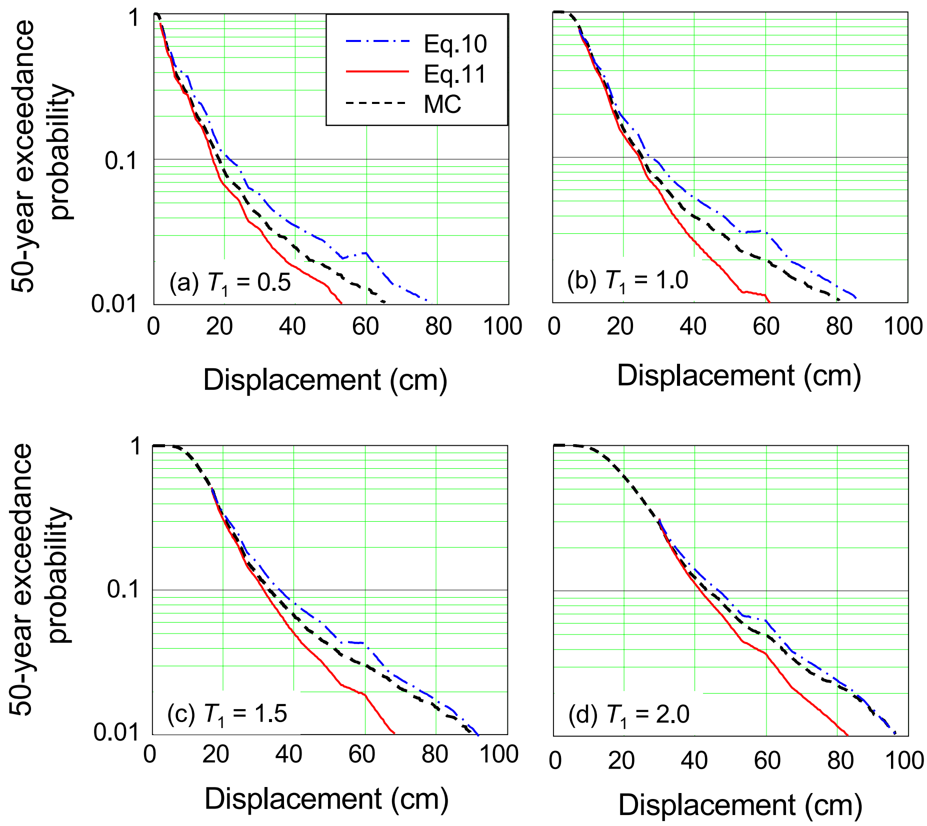


Figure 6. Maximum displacement of inelastic oscillator evaluated by the proposed method

## Conclusions

It is shown in this paper that the spectral displacement of an inelastic oscillator is overly estimated with the use of a UHS along with equivalent linearization technique since the correlation among spectral period is not considered in a UHS. An approximation method is then proposed to estimate the probability distribution function of the maximum response via an equivalent linearization technique considering the probability that the demand spectrum is above the capacity spectrum at two or three spectral periods including the elastic natural period of the oscillator. Using numerical example, it is demonstrated that the exceedance probability of maximum displacement can be evaluated approximately by the proposed method.

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