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MODIFICATIONS OF INTEGRATION ALGORITHMS TO ACCOUNT FOR LOAD DISCONTINUITY IN PSEUDODYNAMIC TESTING

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ABSTRACT

When there is a sudden change in the loading (e.g., rectangular pulse), the discretized version of the load history will involve an artificial impulse, which manifests itself as an amplitude distortion in the structural response obtained by the numerical solution of the equation of motion. An approach to account for load discontinuity by modifying existing integration algorithms used in the solution of force equation of motion is introduced in this paper. Modified versions of four different algorithms, namely Central Difference, Newmark Explicit, α-method with a fixed number of iterations, and Rosenbrock-W integration algorithms are presented. The general approach in modifying an integration algorithm to account for load discontinuity is discussed and the improved accuracy of these modified algorithms is presented through numerical simulations.

Introduction

Pseudodynamic (PSD) test method is a displacement based experimental technique that can be used to determine the behavior of structural systems subjected to dynamic loading. In a PSD test, a direct step by step integration algorithm generates the command displacements by solving the force equation of motion. These displacements are imposed on the test structure by a servo-hydraulic system, and using the measured restoring force feedback from the deformed test structure, the integration algorithm computes the subsequent command displacements. For load-rate insensitive structures, PSD testing method can be applied in slow time (using an expanded time axis), or for structures that exhibit load-rate dependent vibration characteristics, it can be applied at fast rates (ideally in real-time). Both the slow-time and real-time PSD testing have been successfully applied for seismic loading (Mahin 1985, Nakashima 1999), but if the loading history has a sharp discontinuity as in the case of pulse loading (see Fig. 1-a), the numerical solution of the force equation of motion will have an amplitude distortion which may render the PSD test results inaccurate. This is due to the extra impulse (shaded area in Fig. 1-b) introduced during the discretization of the loading history.

To circumvent this problem, the use of step by step solution of the momentum equation of motion was suggested and the resulting improved accuracy was verified through numerical simulations (Chang, 2001, 2002, 2007a) and experiments (Chang, 1998). In this approach, the

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force equation of motion is replaced by its integral form, which is the momentum equation of motion. As a result of the time integration of the load that appears on the right hand side of the momentum equation, provided that the area under the load history is computed correctly, the discontinuity in the load history is eliminated. Although the success of the momentum approach has been presented for Newmark explicit integration algorithm, replacing the force equilibrium equation with the momentum equation may not be a trivial task if one wishes to use an integration algorithm customized (e.g., unconditionally stable, implicit, real-time compatible) for particular testing needs. Other than the momentum approach, in an attempt to obtain an accurate solution in the presence of load discontinuity, Chang, (2007b) also proposed the use of a single time step immediately after the discontinuity that is much smaller than the discretization step size. This small step was recommended to be one-hundredth of the discretization step size or smaller. Especially within the context of real-time PSD testing, a variable time step is detrimental for it would result in inaccurate velocities as the displacement commands are typically imposed using a digital controller with a constant clock speed.

The study presented here introduces an approach, which is referred to as limit approach, to account for the load discontinuity by modifying a given integration algorithm that solves the force equation of motion in its final form. Depending on the way a particular integration algorithm is formulated, these modifications generally involve updated force and/or acceleration values at the time of discontinuity. In the paper, the general approach (i.e., the limit approach) that introduces modifications to a given integration algorithm is discussed and then implemented to derive modified versions of the Central difference, Newmark explicit, α-method with a fixed number of iterations, and Rosenbrock-W integration algorithms. Both α-method and Rosenbrock-W algorithms are suitable for real-time testing, where the former is an implicit scheme. For each of the four integration algorithms considered in this paper, a summary of the original formulation is provided together with its modified version. The improved accuracy of the modified algorithms is presented through numerical simulations.

The Limit Approach

The limit approach starts by defining an intermediate step j just after the load discontinuity between times i (where the discontinuity takes place) and $i+1$ (see Fig. 1-c). In order to be able to set it apart from the discretization time step size of Δt and thereby make the implementation of the limit approach for the modification of an algorithm easier to follow, the time step size associated with step j is identified as $\Delta t'$. It should be noted the load value p_j is the value of the load at the lower end of the discontinuity at step i. From the original formulation of a given integration algorithm, the information for step j (which may include the displacement (u), velocity ($\dot{u}$) and/ or acceleration ($\ddot{u}$), or an intermediate quantity defined by the particular integration algorithm to march forward) can be obtained using the information from step i (see Fig. 1-c) and considering p_j and $\Delta t'$. In order to obtain the information at the lower end of the discontinuity (i^*) and thereby account for discontinuity effects properly, next step involves taking the limit of the expressions that define step j information where $\Delta t'$ goes to zero. On Fig. 1-d the information (i.e., the expressions for the load, displacement etc.) associated with i^* are the results of this limit process. In programming the resulting modified algorithm, a flag needs to be set in order to identify the time step when discontinuity (i.e., step i) takes place. When that happens, the numerical values of for p_{i^*}, u_{i^*} etc. need to be evaluated from the expressions obtained by the limit approach, and using these, the integration algorithm in its original form can march forward to compute information for step $i+1$.

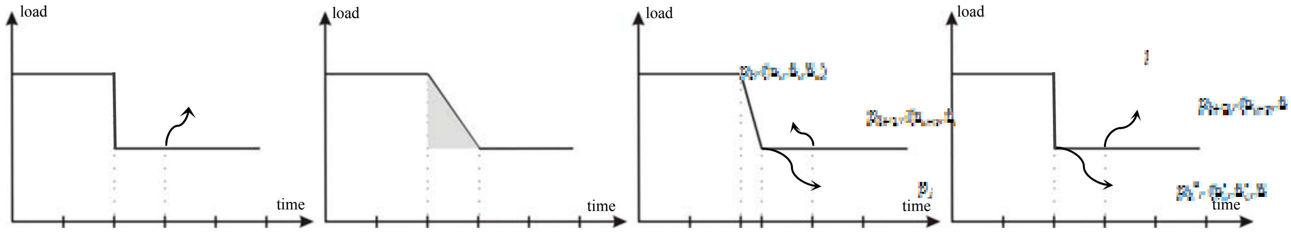

Fig. 1. (a) Load history with discontinuity, (b) Discretized load history, (c) Load history with step j, (d) Load history after performing the limit

Implementation of the Limit Approach

Central Difference Method

The discretized form of the equation of motion for a single-degree-of-freedom (SDOF) system at time step i is:

$$m\,\ddot{u}_i + c\,\dot{u}_i + k\,u_i = p_i \tag{1}$$

Central difference method uses a finite difference approximation for velocity and acceleration (Chopra, 2007). With a constant time step size of Δt, the velocity and acceleration at step i are expressed as:

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \tag{2}$$

$$\ddot{u}_i = \frac{1}{\Delta t}\left(\frac{u_{i+1} - u_i}{\Delta t} - \frac{u_i - u_{i-1}}{\Delta t}\right) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta t^2} \tag{3}$$

Substituting the velocity and acceleration from Eqs. (2) and (3) in Eq. (1), and solving the expression for u_{i+1}:

$$u_{i+1} = \underbrace{\left[p_i - \left(\frac{m}{\Delta t^2} - \frac{c}{2\,\Delta t}\right)u_{i-1} - \left(k - \frac{2\,m}{\Delta t^2}\right)u_i\right]}_{\hat{p}_i} \Big/ \underbrace{\left[\frac{m}{\Delta t^2} + \frac{c}{2\,\Delta t}\right]}_{\hat{k}} \tag{4}$$

Implementation of Limit Approach to Obtain (i^*) Information

Considering Fig. 1-c and rewriting Eqs. (2) and (3) for step i, velocity and acceleration at that step are:

$$\dot{u}_i = \frac{u_j - u_{i-1}}{\Delta t + \Delta t'}, \qquad \ddot{u}_i = \frac{1}{\frac{1}{2}(\Delta t + \Delta t')}\left(\frac{u_j - u_i}{\Delta t'} - \frac{u_i - u_{i-1}}{\Delta t}\right) \tag{5), (6}$$

Also, using the same equations, velocity and acceleration at step j can be expressed as:

$$\dot{u}_j = \frac{u_{i+1} - u_i}{\Delta t + \Delta t'}, \qquad \ddot{u}_j = \frac{1}{\frac{1}{2}(\Delta t + \Delta t')}\left(\frac{u_{i+1} - u_j}{\Delta t} - \frac{u_j - u_i}{\Delta t'}\right) \tag{7), (8}$$

Eqs. (9) and (10) express the equation of motion at time steps i and j, respectively:

$$m\,\ddot{u}_i + c\,\dot{u}_i + k\,u_i = p_i \tag{9}$$

$$m\,\ddot{u}_j + c\,\dot{u}_j + k\,u_j = p_j \tag{10}$$

Once Eq. (9) is solved for u_j (after substituting for $\dot{u}_i$ and $\ddot{u}_i$ from Eqs. (5) and (6)), the result can be used to replace u_j in Eq.(10) which then can be solved for u_{i+1} (after substituting for $\dot{u}_j$ and $\ddot{u}_j$ from Eqs. (7) and (8)). Performing a limit where $\Delta t' \rightarrow 0$ gives an expression for u_{i+1} in terms of u_{i-1} and u_i:

$$u_{i+1} = \underbrace{\left[\underbrace{\frac{p_i + p_j}{2}}_{p_i^*} - \left(\frac{m}{\Delta t^2} - \frac{c}{2\,\Delta t}\right)u_{i-1} - \left(k - \frac{2\,m}{\Delta t^2}\right)u_i\right]}_{\hat{p}_i^*} \Big/ \underbrace{\left[\frac{m}{\Delta t^2} + \frac{c}{2\,\Delta t}\right]}_{\hat{k}} \tag{11}$$

Comparing Eq. (4) with Eq.(11), application of limit approach to Central difference method reveals that to account for the load discontinuity, only the load value for (i^*) needs to be redefined as the average value of the loads at each end of the discontinuity.

Explicit Newmark method

Newmark family integration methods (Newmark, 1959) can be customized by selecting two parameters (γ and β) which specify the variation of acceleration over a time step. These parameters also determine the accuracy and stability characteristics of the method. For $\gamma = 0.5$ $\beta = 0$ Newmark's method becomes explicit and conditionally stable:

$$u_{i+1} = u_i + \Delta t\,\dot{u}_i + 0.5\,\Delta t^2 \ddot{u}_i \tag{12}$$

$$\ddot{u}_{i+1} = \frac{p_{i+1} - k\,u_{i+1} - c[\dot{u}_i + 0.5\,\Delta t\,\ddot{u}_i]}{m + 0.5\,c\,\Delta t} \tag{13}$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta t[0.5\,\ddot{u}_i + 0.5\,\ddot{u}_{i+1}] \tag{14}$$

Implementation of Limit Approach to Obtain (i^*) Information

Defining the information for step j using Eqs. (12)-(14) as required by the limit approach:

$$u_j = u_i + \Delta t' \, \dot{u}_i + 0.5 \, \Delta t'^2 \ddot{u}_i \tag{15}$$

$$\ddot{u}_j = \frac{p_j - k \, u_j - c[\dot{u}_i + 0.5 \, \Delta t' \, \ddot{u}_i]}{m + 0.5 \, c \, \Delta t'} \tag{16}$$

$$\dot{u}_j = \dot{u}_i + 0.5 \, \Delta t' (\ddot{u}_i + \ddot{u}_j) \tag{17}$$

Performing a limit on the above expressions where $\Delta t' \to 0$, will yield

$$u_i^* = u_i, \;\; \dot{u}_i^* = \dot{u}_i \qquad (18), (19)$$

$$\ddot{u}_i^* = \frac{p_i^* - k \, u_i - c \, \dot{u}_i}{m} \tag{20}$$

where $p_i^* = p_j$

From above equations, it turns out that, while the values for displacement and velocity remain the same, the acceleration value at the lower end of the discontinuity (i^*) needs to be updated. By stepping forward to step $i+1$ using the updated information from Eqs. (18)- (20) derived consistently with Newmark explicit method formulation, the load discontinuity will be taken care of properly.

Rosenbrock

Considering an SDOF system the general implementation of Rosenbrock integration method proposed by Lamarche (2009) for real-time pseudo-dynamic testing is described in Fig. 2.

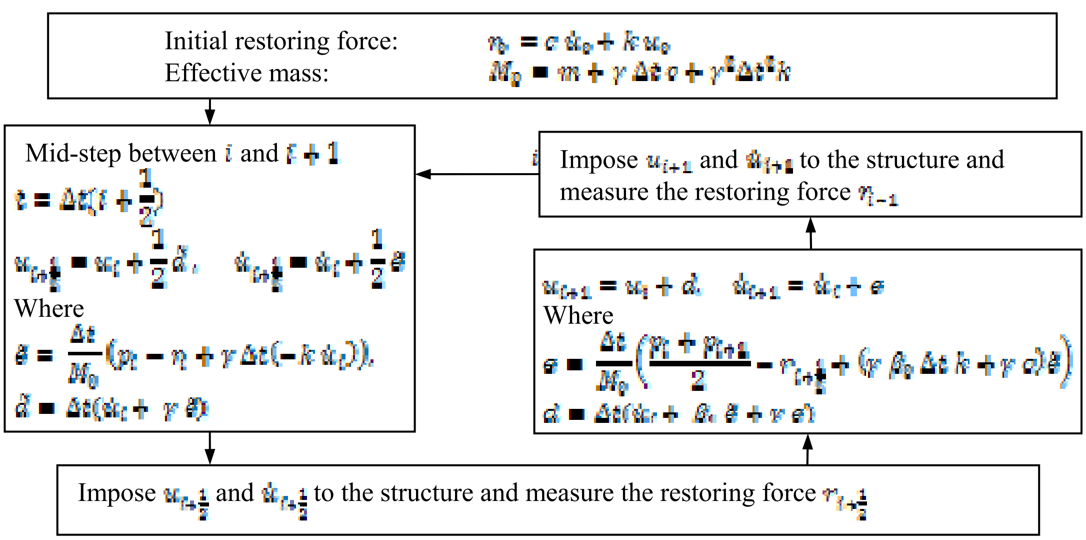

Fig. 2. Rosenbrock integration algorithm

Implementation of Limit Approach to Obtain (i^*) Information

Introducing step j and taking the limit for the Rosenbrock algorithm, it can be easily shown that u_j and $\dot{u}_j$ approach to u_i and $\dot{u}_i$, respectively (or equivalently $u_i{}^*$ and $\dot{u}_i{}^*$ are the same as u_i and $\dot{u}_i$, respectively). As a result, for Rosenbrock algorithm to handle the load discontinuity properly, the information at step $i+1$ right after the discontinuity needs to be calculated using $u_i{}^*$, $\dot{u}_i{}^*$ and $p_i{}^*$ which are the same as u_i, $\dot{u}_i$, and p_j, respectively.

Alpha method

Alpha method (Shing *et al* 2002) is an implicit method used in real-time pseudo-dynamic testing which is based on Hilber α-method (Hilber *et al* 1977). As can be seen from Fig. 3, Alpha method starts by computing a predictor displacement in stage $\boxed{1}$, which is followed by a fixed number of iteration substeps in stages $\boxed{2}$ and $\boxed{3}$; during the last iteration substep an equilibrium correction is performed. Through the equilibrium error correction (stages $\boxed{4}$ and $\boxed{5}$) the displacement and restoring force values are made available for the computation of the next step predictor displacement and, as a result, the actuator moves without interruption.

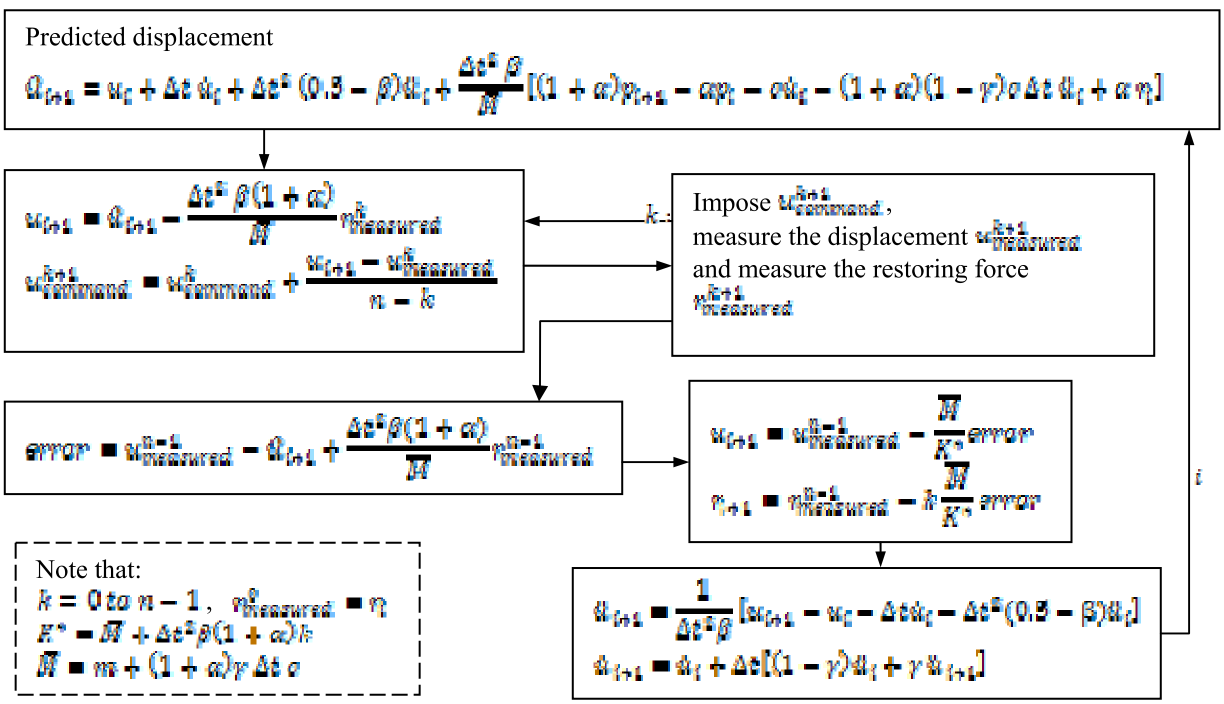

Fig. 3. Alpha method algorithm

Implementation of Limit Approach to Obtain (i^*) Information

Upon the application of the limit approach starting from stage $\boxed{1}$ in Fig. 3, the predicted displacement at step j approaches to u_i after setting $\Delta t' \rightarrow 0$ (or equivalently $\hat{u}_i^* = u_i$) and the displacement term in stage $\boxed{2}$ becomes constant (i.e., $u_i^* = u_i$). Considering that both parameters K^* and $\bar{M}$ approach to m application of the limit to stages $\boxed{4}$ and $\boxed{5}$ gives:

$$error = u^{n-1}_{measured} - u_i \tag{21}$$

$$\lim_{\Delta t' \to 0} u_j = u_i^* = u^{n-1}_{measured} - (u^{n-1}_{measured} - u_i) = u_i \tag{22}$$

$$\lim_{\Delta t' \to 0} r_j = r_i^* = r^{n-1}_{measured} - k\,(u^{n-1}_{measured} - u_i) = r_i \tag{23}$$

Eq. (23) is true for linear elastic systems. In stage $\boxed{6}$ the expression for acceleration can be revised for step j as

$$\ddot{u}_j = \frac{1}{\Delta t'^2 \beta}\left[u_j - u_i - \Delta t' \dot{u}_i - \Delta t'^2 (0.5 - \beta)\ddot{u}_i\right] \tag{24}$$

Upon substituting the expression of u_j from stage $\boxed{2}$ which has the expression for $\hat{u}_j$ embedded from stage $\boxed{1}$, and as a result of the cancellations that take place, the zero over zero indeterminacy as $\Delta t'$ approaches to zero is eliminated; and the acceleration for i^* is obtained:

$$\ddot{u}_i^* = \frac{-p_i\alpha + (1+\alpha)p_i^* - c\,\dot{u}_i - r_i}{m} \tag{25}$$

where $p_i^* = p_j$

And the expression for velocity in stage $\boxed{6}$ yields

$$\dot{u}_i^* = \dot{u}_i \tag{26}$$

The above application of the limit approach in modifying Alpha method reveals that, to compute the information at step $i+1$ right after the discontinuity, information from i^* needs to be used where the updated acceleration from Eq.(26) is used together with the value of the load at the lower end of the discontinuity.

Numerical Simulations

In order to verify the success of the proposed modifications in handling the load discontinuity, numerical simulation results for each integration algorithm are presented here. An undamped linear SDOF system with $m = 0.2533\ kip.sec^2/in$, and $k = 10kips/in$ (i.e.,undamped natural period $T_n = 1\ sec.$) subjected to the two loading cases shown in Fig.4 is considered. Fig. 4 (a) is a step pulse with an amplitude of 10 *kips* and duration of 0.1 *sec*.;

whereas in Fig. 4 (b) the load value changes from +10 to -10 *kips* at the discontinuity and then increases to zero linearly over a duration of 0.1 *sec*.

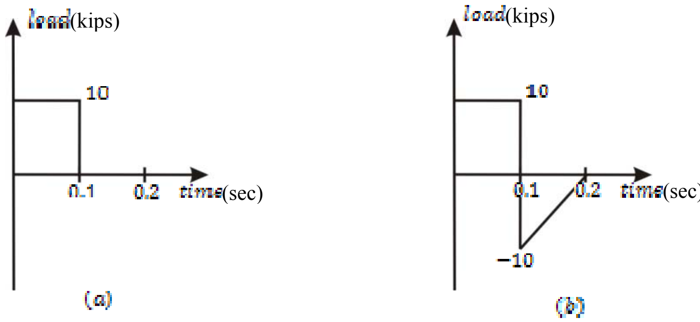

Fig. 4. Load cases with discontinuity

When discretizing the loading history for subsequent use in the step-by-step solution of the equation of motion, selection of a very small time step will minimize the extra impulse and in turn the amplitude distortion introduced in the response. To be able to check the ability of the modified algorithms in handling large extra impulses, the time step size for the numerical simulations was selected as 0.1 *sec.*; which also made the results of this study comparable with that of Chang (2001), where it was shown that time step can be selected as large as the impulse duration when the momentum equation is used. It needs to be pointed out that for Central difference and Newmark explicit algorithms $\frac{\Delta t}{T_n}$=0.1 sets the limit for accuracy, beyond which the amplitude decay and period elongation introduced by the algorithm may be significant (Chopra, 2007).

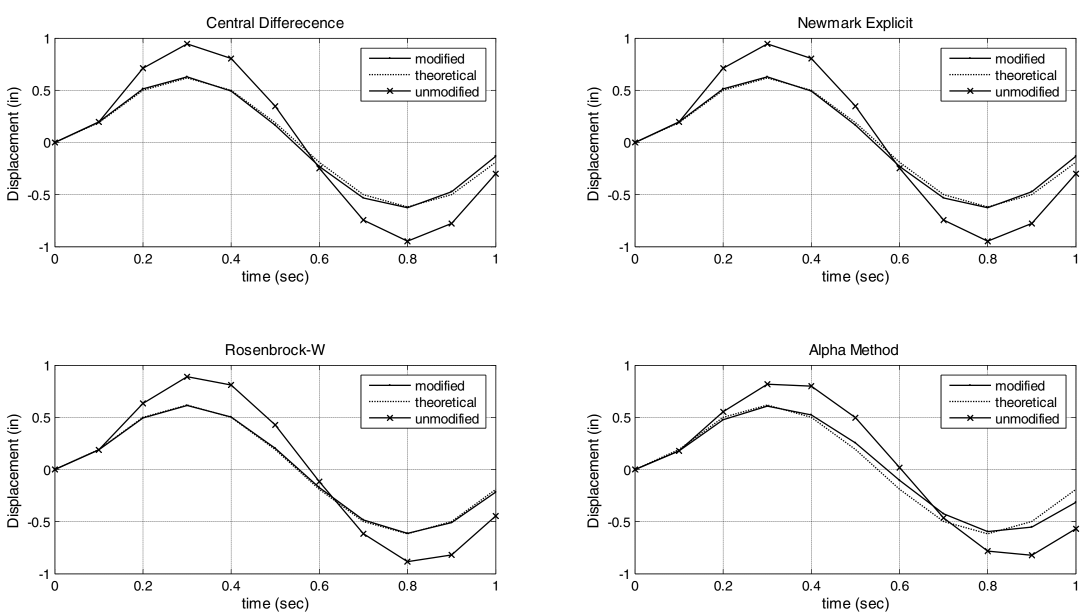

Fig. 5. Simulation results for the loading case in Fig. 4(a)

Fig. 5 compares the theoretical response with the results obtained from the original (unmodified) and modified versions of the integration algorithms considered in this study. Both versions of each algorithm were programmed in MATLAB. In the modified version, a flag was set to identify the discontinuity, and the modified expressions for load and/or acceleration were introduced as derived here. As can be seen from Fig. 5, there is a significant difference between the unmodified and theoretical responses; whereas the modified results are in good agreement with the theoretical response. Similarly, for Fig. 6 which considers nonzero load values after the discontinuity, the proposed modifications provide considerable improvements in the accuracy of the numerical results. The unmodified responses from each algorithm between Fig. 5 and 6 are the same as expected, since with a time step size of 0.1 *sec.*, the discretized versions of the load cases in Fig. 4 (a) and (b) are the same (see Fig. 1 (b)). Depending on the accuracy characteristics of the particular integration algorithm, the agreement between the theoretical response and the modified numerical solution can be improved by using a smaller time step.

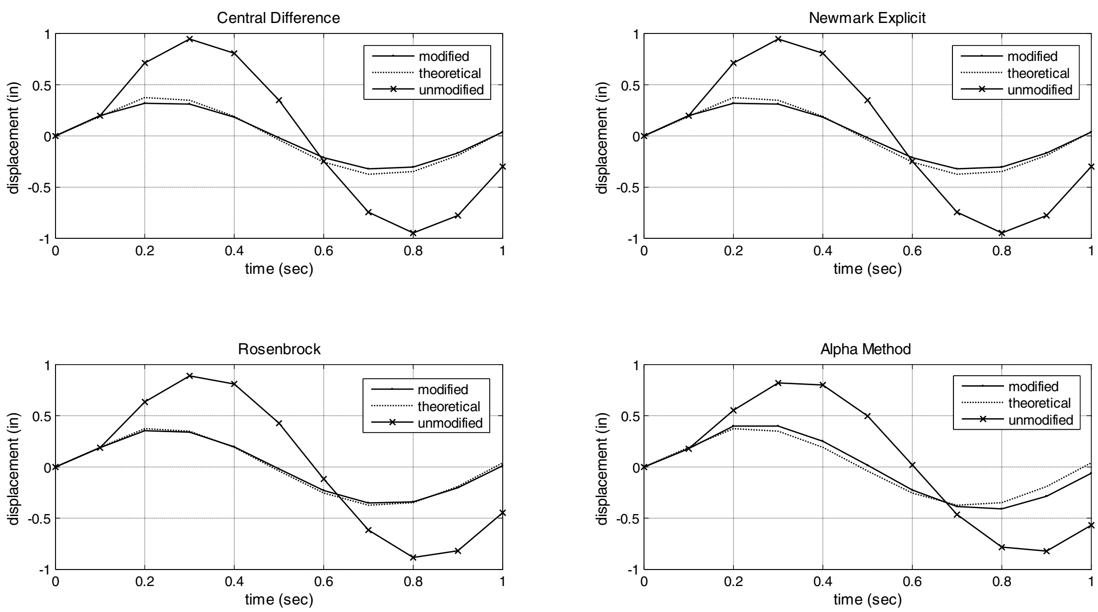

Fig. 6. Simulation results for the loading case in Fig. 4(b).

Conclusion

Pseudodynamic testing method has been implemented successfully both in slow and real-time for seismic loading of structures. However, when a sharp discontinuity exists in the loading history as in the case of pulse loading, the discretized version of the load will have an extra distortion which manifests itself as an amplitude distortion in the numerical response and may render the pseudodynamic test results inaccurate. Other than using very small time steps in discretizing the load, previous studies proposed the use of numerical solution of the momentum equation of motion that replaces the force equation of motion. Although the success of the momentum approach has been presented using Newmark explicit integration algorithm, replacing the force equilibrium with the momentum equation may not be a simple task if one wishes to use an integration algorithm customized for particular testing needs. The study presented here introduces a limit approach that considers the force equation of motion and modifies the integration algorithms in their final form to account for the load discontinuity. The general modification approach and its implementation to four integration algorithms are provided together with numerical simulation results that show the improved accuracy of the modified algorithms.

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